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A NOTE ON HAMILTON'S EQUATIONS AND INVARIANT IMBEDDING

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PREFACE

This Memorandum results from RAND's continuing study of computational aspects of the solution of the equations of mathematical physics. A new basic partial differential equation of particle mechanics is derived.

SUMMARY

Consider the motion of a particle on a line where, as usual, we characterize the process by a Hamiltonian $H = H(q,p)$. The equations of motion are

(1) $\dot{q} = H_p,$

(2) $-\dot{p} = H_q, \quad 0 \leq t \leq T.$

As boundary conditions we specify

(3) $q(0) = 0,$

(4) $p(T) = c.$

If our aim is to solve the system of equations (1) and (2), subject to the boundary conditions in (3) and (4), using a digital computer, we face the well-known difficulties of solving nonlinear two-point boundary-value problems.

It is tempting to try to determine the unknown displacement at time T, $q(T)$, so that we shall have a complete set of conditions at time T. This would enable us to integrate the system (1) and (2) numerically, subject to initial values. Toward this end we introduce the function

(5) $r(c,T)$ = the displacement at time T, the initial displacement being zero and the momentum at time T being c.

We derive a partial differential equation for this function.

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A NOTE ON HAMILTON'S EQUATIONS AND INVARIANT IMBEDDING

1. INTRODUCTION

Consider the motion of a particle on a line where, as usual, we characterize the process by a Hamiltonian $H = H(q,p)$. The equations of motion are

$$(1.1) \qquad \dot{q} = H_p,$$

$$(1.2) \qquad -\dot{p} = H_q, \quad 0 \leq t \leq T.$$

As boundary conditions we specify

$$(1.3) \qquad q(0) = 0,$$

$$(1.4) \qquad p(T) = c.$$

If our aim is to solve the system of equations (1.1) and (1.2), subject to the boundary conditions in (1.3) and (1.4), using a digital computer, we face the well-known difficulties of solving nonlinear two-point boundary-value problems.

It is tempting to try to determine the unknown displacement at time T, $q(T)$, so that we shall have a complete set of conditions at time T. This would enable us to integrate the system (1.1) and (1.2) numerically, subject to initial values. Toward this end we introduce the function

(1.5) $\quad r(c,T)$ = the displacement at time T, the initial displacement being zero and the momentum at time T being c.

We wish to derive an equation for $r(c,T)$.

2. INVARIANT IMBEDDING

In earlier papers [1,2], we showed the following:

If

$$\frac{du}{dz} = F(u,v), \qquad u(0) = 0, \tag{2.1}$$

$$-\frac{dv}{dz} = G(u,v), \qquad v(x) = c, \tag{2.2}$$

for

$$0 \leq z \leq x, \tag{2.3}$$

and if

$$r(c,x) = u(z)\Big|_{z=x}, \tag{2.4}$$

then the function $r(c,x)$ *satisfies the partial differential equation*

$$r_x = F(r,c) + G(r,c)r_c \tag{2.5}$$

and the initial condition

$$r(c,0) = 0. \tag{2.6}$$

Upon applying this result to the system of equations in (1.1) and (1.2), we find that the function $r(c,T)$ satisfies the equation

$$\frac{\partial r}{\partial T} = H_p(r,c) + H_q(r,c)\,\frac{\partial r}{\partial c}, \tag{2.7}$$

or, more elegantly,

$$\frac{\partial r}{\partial T} = \frac{\partial H(r,c)}{\partial c}. \tag{2.8}$$

This is the desired result, apparently a new equation of mechanics.

3. AN EXAMPLE

Consider the case of harmonic oscillations for which

$$H(q,p) = \frac{p^2}{2m} + \frac{1}{2}\,kq^2, \tag{3.1}$$

and

$$q(0) = 0, \quad r(T) = c. \tag{3.2}$$

For the unknown displacement at time T, $r(c,T)$, equation (2.8) becomes

$$\frac{\partial r}{\partial T} = \frac{\partial}{\partial c}\left[\frac{c^2}{2m} + \frac{1}{2}\,kr^2\right] = \frac{c}{m} + kr\,\frac{\partial r}{\partial c}. \tag{3.3}$$

The solution of this equation, subject to the condition (2.6) is

$$(3.4) \qquad r(c,T) = \frac{c}{\sqrt{km}} \tan \left(\sqrt{\frac{k}{m}}\, T\right).$$

This provides the desired displacement at time T and also shows that the function $r(c,T)$ may become infinite for a finite value of T.

4. DISCUSSION

The previous discussion may be generalized to the case of many particles moving in three-dimensional space using the theorem in [1]. Furthermore, various combinations of the q's and p's may be specified at the ends. There are immediate applications to perturbation analysis of results of this nature; see [3].

REFERENCES

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